

On the large Ω -deformations in the Nekrasov-Shatashvili limit of $\mathcal{N} = 2^*$ SYM

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ABSTRACT: We study the multi-instanton partition functions of the Ω -deformed $\mathcal{N} = 2^*$ SU(2) gauge theory in the Nekrasov-Shatashvili (NS) limit. They depend on the deformation parameters ϵ_1 , the scalar field expectation value a , and the hypermultiplet mass m . At fixed instanton number k , they are rational functions of ϵ_1, a, m and we look for possible regularities that admit a parametrical description in the number of instantons. In each instanton sector, the contribution to the deformed Nekrasov prepotential has poles for large deformation parameters. To clarify the properties of these singularities we exploit Bethe/gauge correspondence and examine the special ratios m/ϵ_1 at which the associated spectral problem is n -gap. At these special points we illustrate several structural simplifications occurring in the partition functions. After discussing various tools to compute the prepotential, we analyze the non-perturbative corrections up to $k = 24$ instantons and present various closed expressions for the coefficients of the singular terms. Both the regular and singular parts of the prepotential are resummed over all instantons and compared successfully with the exact prediction from the spectral theory of the Lamé equation, showing that the pole singularities are an artifact of the instanton expansion. The analysis is fully worked out in the 1-gap case, but the final pole cancellation is proved for a generic ratio m/ϵ_1 relating it to the gap width of the Lamé equation.

KEYWORDS: Extended Supersymmetry, Nonperturbative Effects, Supersymmetric Effective Theories

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1 Introduction

Four-dimensional gauge theories with rigid $\mathcal{N} = 2$ supersymmetry play a central role in modern theoretical physics. Roughly speaking, they stand between the *real* world of phenomenological applications of softly broken $\mathcal{N} = 1$ models, and the highbrow realm of maximal $\mathcal{N} = 4$ theories. The theoretical control over $\mathcal{N} = 2$ super Yang-Mills (SYM) theories is largely based on the key fact that their low-energy effective physics is captured by the Seiberg-Witten (SW) curve. It describes non-perturbatively the geometry of the moduli space of the gauge theory vacua [1, 2]. In the simplest case when the gauge group is $SU(2)$, the SW curve is a complex torus whose modular parameter τ is the complexified gauge coupling constant at low energy. Quite generally, the effective action has 1-loop perturbative corrections and instanton non-perturbative corrections. It may be fully represented in terms of an analytic function, the prepotential \mathcal{F} . It depends on the vacuum

expectation value $a = (a_1, a_2, \dots)$ of the scalars in the adjoint gauge multiplet and on the masses $m = (m_1, m_2, \dots)$ of the hypermultiplets [3]. More structure is provided by combining $\mathcal{N} = 2$ supersymmetry with conformal invariance. Here, we shall focus on a standard benchmark, i.e. the so-called $\mathcal{N} = 2^*$ theory whose matter content is an adjoint massive hypermultiplet. The gauge coupling runs at 1-loop by perturbative corrections proportional to the hypermultiplet masses. The most interesting effects are due to the instantons and are of course non-perturbative. Our analysis will be aimed at studying of these contributions leaving aside the perturbative part.

The non-perturbative corrections are predicted correctly by the SW curve, but may be computed in an alternative and somehow more powerful way by means of localization [4, 5]. The $\mathcal{N} = 2$ partition function is topologically twisted and localized on the multi-instanton moduli spaces. These are finite-dimensional spaces, but nevertheless are plagued by divergences that appear in the UV at small instanton size and also in the IR when the instanton centers are moved to infinity. These problems have been cleverly by-passed by considering the gauge theory in a curved background, known as Ω -background [4–6]. This construction introduces two deformation parameters ϵ_1, ϵ_2 that break 4d Poincaré invariance but fully regularize the moduli space integration [7–14].¹ The resulting Nekrasov instanton partition function $Z_{\text{inst}}(\epsilon_1, \epsilon_2, a, m)$ defines a non-perturbative ϵ -deformed prepotential by means of the relation

$$F_{\text{inst}}(\epsilon_1, \epsilon_2, a, m) = -\epsilon_1 \epsilon_2 \log Z_{\text{inst}}(\epsilon_1, \epsilon_2, a, m). \quad (1.1)$$

The SYM prepotential $\mathcal{F}(a, m)$ is recovered in the $\epsilon_1, \epsilon_2 \rightarrow 0$ limit, up to the classical tree-level contribution.

There are several reasons to regard the Ω -deformation as more than a mere regularization. i.e. to study the properties of (1.1) at finite deformation parameters ϵ_1, ϵ_2 . One string motivated analysis examines systematically the expansion of (1.1) around the undeformed point $\epsilon_1 = \epsilon_2 = 0$.² Another option is to look at the exact dependence of (1.1) on its parameters, including ϵ_1, ϵ_2 . Ideally, one would like to emphasize special features appearing order by order in the instanton expansion. In this perspective, a major simplification occurs in the Nekrasov-Shatashvili (NS) limit [31] where one of the two ϵ parameters vanishes, say $\epsilon_2 = 0$. The resulting theory has a 2d $\mathcal{N} = 2$ super-Poincaré invariance. The supersymmetric vacua of this gauge theory are the eigenstates of the quantum integrable system obtained by quantization of the classical integrable system associated with the geometry of the moduli space of undeformed $\mathcal{N} = 2$ theory. Under this Bethe/gauge map,

¹Quite remarkably, the Ω -background is equivalent to a supergravity background with a non-trivial graviphoton field strength, on the instanton moduli space [15, 16]. For the string interpretation of the Ω -background and its BPS excitations see [17–19].

²Indeed, it is convenient to add to (1.1) the perturbative part F_{pert} and define the amplitudes $F^{(n,g)}$ from the double expansion $F_{\text{pert}} + F_{\text{inst}} = \sum_{n,g=0}^{\infty} F^{(n,g)} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^g$. The amplitudes $F^{(0,g)}$ with $g \geq 1$ correspond to F-terms in the effective action of the form \mathcal{W}^{2g} , where \mathcal{W} is the chiral Weyl superfield containing the graviphoton field strength. These terms may also be obtained from the genus g partition function of the $\mathcal{N} = 2$ topological string [20] and satisfy a holomorphic anomaly equation [21–24]. Similar interpretations can be extended to the amplitudes $F^{(n,g)}$ with $n \neq 0$ [25–29], see also [30].

the deformation parameter ϵ_1 plays the role of the Planck constant. In the NS limit, saddle point methods permit to derive a generalized SW curve [32, 33].³

Another pivotal tool to control the generalized prepotential (1.1) non-perturbatively in ϵ_1, ϵ_2 is the AGT correspondence [39] that maps ϵ -deformed $\mathcal{N} = 2$ instanton partition functions to conformal blocks of a suitable CFT. The details of the gauge theory (gauge group and matter content) determine the worldsheet genus and the number and conformal weights of the insertions in the relevant conformal block. The AGT correspondence can be checked by comparing the expansion in instanton number with an expansion in a complex structure parameter of the corresponding punctured Riemann surface on the CFT side [40–42]. In the specific case of the $\mathcal{N} = 2^*$ ϵ -deformed theory with gauge group $SU(2)$, the relevant CFT object is the one-point conformal block on the torus. An important suggestion from the AGT correspondence is that the conformal blocks have nice modular properties with a possible counterpart in the gauge theory as a generalized S -duality at non zero ϵ_1, ϵ_2 .⁴ Indeed, modular transformation properties are well known in the undeformed case [45, 46] and may be explored in the deformed case [47, 48]. For conformal gauge theories with a single gauge group, such as $SU(2)$ theory with $N_f = 4$ and the $\mathcal{N} = 2^*$ theory, it has been possible to make a lot of progress in resumming the instanton expansion by expressing the (deformed) prepotential in terms of quasi-modular functions (Eisenstein series) of the instanton coupling constant. This has been done both from the gauge theory perspective [49–55] and, exploiting AGT, on the CFT side [56–59]. An alternative route exploits an exact WKB analysis [35, 36, 60–63].⁵

Despite being a powerful tool, quasi-modular expansions are commonly studied after expanding the prepotential in inverse powers of (the components of) a , a natural setup from the point of view of the SW curve. As a general feature, the coefficient of each power of $1/a$ turns out to be a polynomial in Eisenstein series and θ -functions with definite modular properties that reflect the $SL(2, \mathbb{Z})$ symmetry of the high energy and extend S -duality to the effective theory. These coefficients have also a polynomial dependence on ϵ_1, ϵ_2 and the hypermultiplet masses m . So, in a sense, this large a expansion amounts to an educated reshuffling of the small ϵ_1, ϵ_2 expansion. Instead, at finite parameters $\epsilon_1, \epsilon_2, a, m$ the Nekrasov partition functions Z_k at instanton number k is computable as a rational functions of them. On one hand, this means that there is a lot of structure to be revealed and understood. On the other hand, it seems hopeless to look for regularities at increasing k due to the growing complexity of Z_k .

At least in the NS limit, these difficulties may be overcome by tuning the hypermultiplet mass m and taking it to be proportional to ϵ_1 with definite special ratios $\frac{m}{\epsilon_1} = n + \frac{1}{2}$, where $n \in \mathbb{N}$. In the $\mathcal{N} = 2^*$ theory, these points are the n -gap cases of the associated spectral problem predicted by the Bethe/gauge correspondence, i.e. the elliptic Calogero-Moser

³A matrix model approach to deformed SW theory is developed in [34–38].

⁴Notice that the SW contour integral methods remain valid also when both ϵ_1 and ϵ_2 are non-vanishing [43, 44].

⁵The WKB analysis is also interesting because it allows to access non-perturbative aspects of the ϵ_1 expansion, see for instance [64–68].

system [31].⁶ At these special points, we shall discuss a remarkable simplification in the functions Z_k . In the simplest 1-gap case, they can be reduced to rational functions of a single scaled variable and their denominator is a simple power of the factor $4a^2 - \epsilon_1^2$. Thus the complexity of the computation is definitely comparable with the large a approach where modularity was discovered. Besides, at the technical level, the functions Z_k may be obtained by expanding the eigenvalues of a Lamé equation in terms of its Floquet exponent, the quasi-momentum.⁷

Apart from the general Bethe/gauge framework leading to the calculation of Z_k , the interesting goal is to identify which features of Z_k may be described parametrically in k , i.e. at all instanton numbers. Our investigation will show that a special role is played by the pole singularities of Z_k in the apparently singular *large* Ω -deformation $\epsilon_1 \rightarrow 2a$. It is not easy to understand these poles from the gauge theory point of view. In the S^4 partition function, for example, a/ϵ is necessarily taken to be imaginary so $4a^2 - \epsilon_1^2$ is never zero. However, if we forget about the reality properties of a or ϵ , then indeed the Nekrasov partition function Z_k have pole singularities at the above points. To give an interpretation, some hints come from the analysis of the 5d origin of these poles in the Ω -deformation construction [10, 69]. Basically, the instanton partition functions at a certain instanton number k may be computed in a reduced supersymmetric quantum mechanical model with hamiltonian H_k . They arise in the 4d $\beta \rightarrow 0$ limit of twisted partition functions $\text{Tr}(-1)^F e^{-\beta H_k} e^{\beta \Gamma} = \text{Tr}_{\mathcal{H}_{\text{susy}}^{(k)}} (-1)^F e^{\beta \Gamma}$, where $\mathcal{H}_{\text{susy}}$ is the space of supersymmetric states and β is the radius in the 5d space $S^1_\beta \times \mathcal{M}^4$. The term Γ is a linear combination of a and ϵ_1 . The poles in Z_k appear when $\Gamma = 0$ and signal the infinite dimension of the set of supersymmetric states. Their contribution is regularized by the Γ term leading to finite sums of the form $\sum_n e^{-\beta n(c\epsilon_1 + c'a)}$ with n over some subset of the integers.⁸ In our setup, $\Gamma = 0$ precisely when $\epsilon_1 \pm 2a$ vanishes. This specific origin of the poles suggests that they may be simple enough to be described at all k . So, in summary, our admittedly vague claim is that

The “large” ϵ_1 singularities of the k -instanton partition functions are closely related to the structure of $\mathcal{H}_{\text{susy}}^{(k)}$ and could feature a particularly simple dependence on the instanton number k .

In this paper, we shall explore this idea in more quantitative terms. To this aim, in section 2, we shall begin by reviewing some special features of the Nekrasov partition functions in

⁶It is interesting to note that the finite-gap cases of Lamé equation, extensively investigated in this paper, from the Liouville theory point of view exactly correspond to torus 1-point insertion of the n -th level degenerate field $\varphi_{n,1}$.

⁷This construction has an AGT counterpart as a modular expansion of the conformal blocks computed by null vector decoupling equations.

⁸The simplest example is that of supersymmetric quantum mechanics on $\mathbb{C}^2 = \{(u, v), u, v \in \mathbb{C}\}$. If this system has global symmetry generators (J_1, J_2) being $(1, 0)$ on u and $(0, 1)$ on v , then the partition function $Z(\beta, \epsilon_1, \epsilon_2) = \text{tr}(-1)^F e^{i\beta\epsilon_1 J_1} e^{i\beta\epsilon_2 J_2}$ reduces to a sum over $\mathcal{H}_{\text{susy}} = \oplus_{m,n \geq 0} \mathbb{C} z^m w^n$ that gives $Z(\beta, \epsilon_1, \epsilon_2) = (1 - e^{i\beta\epsilon_1})^{-1} (1 - e^{i\beta\epsilon_2})^{-1}$. In this case, the singularity is at small $\epsilon_1, \epsilon_2 \rightarrow 0$. In the gauge theory case, we need to add the scalar field vacuum expectation values a playing formally a role similar to that of ϵ_1, ϵ_2 [70].

the n -gap NS limit by analyzing explicit examples at low instanton number. In section 3, we shall review the Bethe/gauge connection for the $\mathcal{N} = 2^*$ theory. In particular, we shall illustrate the relation between the Nekrasov functions and the exact properties of the Lamé equation, as well as various algorithms for the actual computation of Z_k for large k . This relatively simple technology will be exploited in section 5 where we shall analyze the explicit data for Z_k at the 1-gap point considering up to $k = 24$ instantons. The main outcome will be a set of closed expressions providing the parametric dependence on k of the poles of Z_k as $\epsilon_1 \rightarrow 2a$. This will be the quantitative version of the above claim. Besides, these singular contributions admit a resummation at all instantons which turns out to be *regular* at the naively singular point. This finiteness property has a very simple interpretation in terms of the spectral properties of the Lamé equation. The same study is then extended to the 2-gap case and generalized in section 6 to the generic μ case. Remarkably, the finite resummation of the Nekrasov poles turns out to be closely related to the Lamé equation gap widths.

2 Structure of Nekrasov instanton partition functions

The grand-canonical instanton partition function in the Ω -deformed $\mathcal{N} = 2^*$ SU(2) gauge theory may be written as an expansion in the number of instantons according to

$$Z_{\text{inst}}(\epsilon_1, \epsilon_2, a, m) = 1 + \sum_{k=1}^{\infty} Z_k(\epsilon_1, \epsilon_2, a, m) q^{2k}, \quad (2.1)$$

where $q = e^{i\pi\tau}$ in terms of the complexified gauge coupling τ . The associated deformed prepotential is, see (1.1),

$$F_{\text{inst}}(\epsilon_1, \epsilon_2, a, m) = -\epsilon_1 \epsilon_2 \log Z_{\text{inst}} = \sum_{k=1}^{\infty} F_k(\epsilon_1, \epsilon_2, a, m) q^{2k}, \quad (2.2)$$

where a is the scalar field vacuum expectation value and m is the hypermultiplet mass, as discussed in the Introduction. The explicit function Z_k may be computed by Nekrasov formula [5] encoding localization. In general, for a classical gauge algebra $\mathfrak{g} \in \{A_r, B_r, C_r, D_r\}$ one can efficiently apply the methods in [4, 5, 8, 11, 71–73] to compute (2.2), order by order in the instanton number k . This is straightforward, but computationally quite involved as k increases, limiting the technique to the first few values of k . For our purposes, it is instructive to look at the explicit contributions F_k . At one instanton level, we have the well-known result

$$F_1(\epsilon_1, \epsilon_2, a, m) = -\frac{(4m^2 - (\epsilon_1 - \epsilon_2)^2)(16a^2 - 4m^2 - 3(\epsilon_1 + \epsilon_2)^2)}{8(4a^2 - (\epsilon_1 + \epsilon_2)^2)}. \quad (2.3)$$

At two instanton level, the expression for $Z_2(\epsilon_1, \epsilon_2, a, m)$ is again a rational function of its parameters, but much more involved. Apparently, it is hopeless to look for results parametric in the instanton number k . Going to the Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$ improves little. Just to give a feeling, at $k = 2$ we obtain the ugly combination

$$F_2(\epsilon_1, 0, a, m) = \frac{4m^2 - \epsilon_1^2}{1024(4a^2 - \epsilon_1^2)^3(a^2 - \epsilon_1^2)} \left[-256a^2(192a^6 - 96a^4m^2 + 48a^2m^4 - 5m^6) \right]$$

$$+64 \epsilon_1^2 (1248a^6 - 288a^4m^2 + 81a^2m^4 + 7m^6) - 48 \epsilon_1^4 (848a^4 - 101a^2m^2 + 23m^4) \\ + 4 \epsilon_1^6 (2107a^2 - 75m^2) - 631 \epsilon_1^8 \Big], \quad (2.4)$$

and again it seems very complicated to generalize to higher k . Obviously, the main complications is the increasing complexity of the involved rational functions. In particular, denominators of F_k become more and more complicated as k grows, although always in factorized form as in the above examples. In the following, to simplify the discussion, we shall name *Nekrasov functions* the contributions F_k . Also, we shall name *Nekrasov poles* the singularities associated with the vanishing factors in the denominators of the Nekrasov functions.

Looking at (2.3) and (2.4), it is clear that important simplifications occur if we assume that $m \sim \epsilon_1$. Motivated by the following discussion, we shall parametrize this limit as

$$m^2 = \left(\mu + \frac{1}{4} \right) \epsilon_1^2. \quad (2.5)$$

Dimensional analysis suggests to introduce the quantity

$$\tilde{F}_k(\nu) = \frac{1}{\epsilon_1^2} F_k \left(\frac{2a}{\nu}, 0, a, \frac{2a}{\nu} \sqrt{\mu + \frac{1}{4}} \right). \quad (2.6)$$

This is a function of the scaling variable ν with an omitted understood dependence on the parameter μ . Again, let us look at the first two instanton numbers. We find

$$\tilde{F}_1(\nu) = \frac{2\mu(\mu - \nu^2 + 1)}{\nu^2 - 1}, \quad (2.7) \\ \tilde{F}_2(\nu) = \frac{\mu \left(\mu^3 (5\nu^2 + 7) - 12\mu^2 (\nu^2 - 1)^2 + 6\mu (\nu^2 - 2) (\nu^2 - 1)^2 - 3 (\nu^2 - 4) (\nu^2 - 1)^3 \right)}{(\nu^2 - 4) (\nu^2 - 1)^3}.$$

A special case appears immediately, i.e. $\mu = 2$. We shall call this point the *1-gap NS* limit for reasons that will appear clearly in our later discussion. Computing also \tilde{F}_k for $k = 3, 4, 5$ at $\mu = 2$, we find the following very simple structure of the Nekrasov functions (Q is polynomial)

$$\tilde{F}_k(\nu) = \frac{Q_k(\nu^2)}{(\nu^2 - 1)^{2k-1}}, \quad \deg Q_k = 2k - 1, \quad (2.8)$$

with the following explicit first cases

$$Q_1(x) = -4(x - 3), \quad Q_2(x) = -2(3x^3 - 21x^2 + 57x - 7), \quad (2.9) \\ Q_3(x) = -\frac{16}{3}(x^5 - 11x^4 + 94x^3 - 226x^2 - 111x - 3), \\ Q_4(x) = -7x^7 + 105x^6 - 1395x^5 + 8909x^4 - 10101x^3 - 32133x^2 - 6353x + 15, \\ Q_5(x) = -\frac{8}{5}(3x^9 - 57x^8 + 1668x^7 - 21532x^6 + 85458x^5 + 89754x^4 \\ - 623356x^3 - 413244x^2 - 36189x - 9).$$

In other words, in the denominators of \tilde{F}_k , all factors but one do cancel and there is only one Nekrasov pole at $\nu^2 = 1$. Thus, the analysis of the instanton contribution reduces to

the study of the polynomials in (2.9). Actually, there is another important feature that can be checked from the explicit expressions (2.9) and that we found to be quite general. This is a simple *selection rule* forbidding even poles around $\nu = 1$. In other words, for $\nu \rightarrow 1$

$$\tilde{F}_k(\nu) = \frac{d_1^{(k)}}{(\nu-1)^{2k-1}} + \frac{0}{(\nu-1)^{2k-2}} + \frac{d_2^{(k)}}{(\nu-1)^{2k-3}} + \frac{0}{(\nu-1)^{2k-4}} + \cdots + \frac{d_k^{(k)}}{\nu-1} + \text{regular}, \quad (2.10)$$

and \tilde{F}_k is fully determined by the $k+1$ rational numbers $\mathbf{d}^{(k)} = \{d_1^{(k)}, \dots, d_{k+1}^{(k)}\}$ appearing in the *exact* decomposition (it includes the regular part)

$$\tilde{F}_k(\nu) = d_{k+1}^{(k)} + \sum_{p=1}^k d_p^{(k)} \left(\frac{1}{(\nu-1)^{2k-2p+1}} - \frac{1}{(\nu+1)^{2k-2p+1}} \right). \quad (2.11)$$

Our aim will be that of analyzing the coefficients $d_p^{(k)}$ in (2.11) by considering explicit data at high k . This can be achieved by exploiting the Bethe/gauge correspondence as a device to boost the computation of the Nekrasov functions.

3 Determination of the 1-gap Nekrasov functions from the Lamé equation

As we mentioned, the determination of \tilde{F}_k from a direct application of Nekrasov formula is not feasible for large k and we heavily exploit Bethe/gauge correspondence. The identification (2.5) is precisely associated with the integrable model description of the NS limit $\epsilon_2 = 0$ in terms of the elliptic Calogero-Moser system. This further reduces to the Lamé equation in the $SU(2)$ case considered here [31]. Also, when $\mu = n(n+1)$ and $n \in \mathbb{N}$, the Lamé potential is finite-gap. The associated spectral problem is

$$\psi''(x) - u(x) \psi(x) = \lambda \psi(x), \quad (3.1)$$

with

$$u(x) = \mu \wp(x; \omega, \omega'). \quad (3.2)$$

The Ansatz $\psi(x) = \exp(\int^x v(x') dx')$ leads to the Miura equation

$$v' + v^2 = u + \lambda. \quad (3.3)$$

The Floquet exponent $i\nu$ is defined as the average of $v(x)$ over a period.⁹ The Bethe/gauge correspondence predicts that the full Nekrasov function F_{inst} is essentially the energy eigenvalue of the auxiliary Schrödinger equation (3.1) expressed in terms of the Floquet exponent $\lambda = \lambda(\nu)$. This relation is in 1-1 correspondence with the AGT evaluation of the conformal block as an expansion in terms of the inverse Shapovalov matrix [35, 36, 74]. Thus, if we evaluate this relation and expand in the modular parameter q we can obtain immediately

⁹We shall consider elliptic functions u doubly periodic in the complex plane. The average will be taken along the real axis with a possible imaginary shift. As long as the integration segment does not cross any pole, the result is independent on this shift.

all the Nekrasov functions at finite a . The 1-gap point $\mu = 2$ is particularly simple because of the simplicity of the exact Floquet exponent. To work out the details, let us begin by denoting

$$\mathbb{K} = \mathbb{K}(m), \quad \mathbb{K}' = \mathbb{K}(1-m), \quad q = \exp\left(-\pi \frac{\mathbb{K}'}{\mathbb{K}}\right). \quad (3.4)$$

Then, for $\mu = 2$, the potential (3.2) can be written in the Lamé form by shifting $x \rightarrow x + i\mathbb{K}'$

$$u(x) = 2 \left[-\frac{1+m}{3} + m \operatorname{sn}^2(x, m) \right], \quad (3.5)$$

To map the period interval $[0, 4\mathbb{K}]$ to $[0, 2\pi]$, it is convenient to introduce the scaled quantities

$$\nu = \bar{\nu} \frac{2\mathbb{K}}{\pi}, \quad \bar{\lambda} = \lambda \left(\frac{\pi}{2\mathbb{K}} \right)^2. \quad (3.6)$$

The *exact* Floquet exponent can be written in terms of the Jacobi Z function by the expression [75]

$$i\bar{\nu} = -i \left[i Z \left(\arcsin \sqrt{\frac{\bar{\lambda} + \frac{m+1}{3}}{m}}, m \right) + \frac{\pi}{2\mathbb{K}} \right]. \quad (3.7)$$

For our purposes, it will be important to exploit the following representation of Z

$$Z(z, m) = \int_0^{\sin z} \frac{1}{\sqrt{1-t^2}} \left(\sqrt{1-mt^2} - \frac{\mathbb{E}}{\mathbb{K} \sqrt{1-mt^2}} \right) dt. \quad (3.8)$$

Separating out real and imaginary parts, we obtain the very useful integral representation of the Floquet exponent

$$i\bar{\nu} = \int_1^{\sqrt{\bar{\lambda} + \frac{m+1}{3}}} \frac{dt}{\sqrt{t^2-m}} \left[\sqrt{t^2-1} + \frac{\mathbb{E}}{\mathbb{K} \sqrt{t^2-1}} \right]. \quad (3.9)$$

The integrand can be expanded in series of m and integrated in terms of elementary functions. The resulting expansions — written for the unbarred quantities ν and λ — turns out to be

$$\begin{aligned} i\nu = & \sqrt{\lambda - \frac{2}{3}} \left[1 + \frac{3(3\lambda-5)}{32(3\lambda-2)(3\lambda+1)} m^2 + \frac{3(3\lambda-5)}{32(3\lambda-2)(3\lambda+1)} m^3 \right. \\ & + \frac{3(19035\lambda^4 - 32535\lambda^3 - 4725\lambda^2 + 5979\lambda + 3670)}{8192(3\lambda-2)^2(3\lambda+1)^3} m^4 \\ & + \frac{3(8667\lambda^4 - 15255\lambda^3 - 1269\lambda^2 + 987\lambda + 2390)}{4096(3\lambda-2)^2(3\lambda+1)^3} m^5 \\ & + \frac{3}{262144(3\lambda-2)^3(3\lambda+1)^5} \left(13701555\lambda^7 - 24815160\lambda^6 - 5143095\lambda^5 \right. \\ & \left. \left. + 5885136\lambda^4 + 7293375\lambda^3 - 68742\lambda^2 - 1192083\lambda - 453050 \right) m^6 + \dots \right]. \end{aligned} \quad (3.10)$$

Notice that this series can be extended with minor effort. Inverting the expansion (3.10), we get a series for the eigenvalue λ as a function of the Floquet exponent

$$\begin{aligned} \lambda = & \frac{2}{3} - \nu^2 + \frac{\nu^2 + 1}{16(1 - \nu^2)} m^2 + \frac{\nu^2 + 1}{16(1 - \nu^2)} m^3 + \frac{-235\nu^6 + 229\nu^4 + 271\nu^2 - 233}{4096(\nu^2 - 1)^3} m^4 \\ & + \frac{-107\nu^6 + 101\nu^4 + 143\nu^2 - 105}{2048(\nu^2 - 1)^3} m^5 \\ & - \frac{6265\nu^{10} - 18253\nu^8 + 8172\nu^6 + 19472\nu^4 - 21613\nu^2 + 6085}{131072(\nu^2 - 1)^5} m^6 + \dots \end{aligned} \quad (3.11)$$

Finally, we replace m by q according to (3.4), and obtain

$$\begin{aligned} \lambda = & \frac{2}{3} - \nu^2 - \frac{16(\nu^2 + 1)}{\nu^2 - 1} q^2 - \frac{16(3\nu^6 + 3\nu^4 - 39\nu^2 + 1)}{(\nu^2 - 1)^3} q^4 \\ & - \frac{64(\nu^{10} + \nu^8 - 74\nu^6 + 206\nu^4 + 121\nu^2 + 1)}{(\nu^2 - 1)^5} q^6 + \dots \end{aligned} \quad (3.12)$$

If we now write (3.12) by separating out the q^0 term

$$\lambda = \frac{2}{3} - \nu^2 + \Lambda(\nu, q), \quad (3.13)$$

we can check that the Bethe/gauge correspondence works perfectly in the following form

$$\sum_{k=1}^{\infty} \tilde{F}_k(\nu) q^{2k} = -\frac{1}{2} \int \frac{dq}{q} \Lambda(\nu, q) + 8 \sum_{k=1}^{\infty} \log(1 - q^{2k}). \quad (3.14)$$

In other words, the expansion (3.12) in rational functions is the generating function of the Nekrasov partition functions $\tilde{F}_k(\nu)$, up to trivial operations. The last term in (3.14) is related to a ν -independent term appearing in the prepotential and proportional to $\log \eta(q)$ where $\eta(q)$ is the Dedekind function, see appendix C. The compact relation (3.14) emphasizes the direct relation that links Λ to the infinite set of Nekrasov functions. Just for the purpose of illustration, we can examine (3.14) at the two instanton level. It reads

$$\begin{aligned} & \tilde{F}_1(\nu) q^2 + \tilde{F}_2(\nu) q^4 + \dots \\ & = \frac{4(\nu^2 + 1)}{\nu^2 - 1} q^2 + \frac{2(3\nu^6 + 3\nu^4 - 39\nu^2 + 1)}{(\nu^2 - 1)^3} q^4 + \dots + 8 \left(-q^2 - \frac{3}{2} q^4 + \dots \right) \\ & = -\frac{4(\nu^2 - 3)}{\nu^2 - 1} q^2 - \frac{2(3\nu^6 - 21\nu^4 + 57\nu^2 - 7)}{(\nu^2 - 1)^3} q^4 + \dots, \end{aligned} \quad (3.15)$$

in full agreement with the expressions of Q_1 and Q_2 in (2.9).

4 Algorithms for the Nekrasov functions at generic μ

The analysis of the 1-gap point $\mu = 2$ is important because the Floquet exponent can be given in closed form according to (3.9). The modular expansion in powers of q is just useful to extract the Nekrasov functions order by order in the instanton expansion. Here, we present two algorithms to determine the Nekrasov functions at generic μ , i.e. at generic ratio m/ϵ_1 between the hypermultiplet mass and the deformation parameter.

4.1 Modular expansion from continued fraction expansions

As very nicely observed in [66] for the Mathieu equation (see also the recent developments [67, 68]), the expansion $\lambda(\nu)$ in (3.12) may be extracted from a known continued fraction expansion of the N -th Lamé eigenvalue in terms of N [76]. This can be used to derive efficiently the expansion (3.12) generating the Nekrasov partition function. Notice that this approach works for generic μ , and not only for $\mu = n(n+1)$, i.e. at the n -gap points. In algorithmic form, the method is as follows. We begin by defining $H = 2L - n(n+1)m$, where n is a real parameter not necessarily integer. We also define the coefficients

$$\begin{aligned}\alpha_p &= \frac{1}{2}(n-2p-1)(n+2p+2)m, \\ \beta_p &= 4p^2(2-m), \\ \gamma_p &= \frac{1}{2}(n-2p+2)(n+2p-1)m.\end{aligned}\tag{4.1}$$

Then, we consider the continued fraction

$$\beta_p - H - \frac{\alpha_{p-1}\gamma_p}{\beta_{p-1} - H - \frac{\alpha_{p-2}\gamma_{p-1}}{\beta_{p-2} - H - \dots}} = \frac{\alpha_p\gamma_{p+1}}{\beta_{p+1} - H - \frac{\alpha_{p+1}\gamma_{p+2}}{\beta_{p+2} - H - \dots}}.\tag{4.2}$$

Expanding L in powers of m with p being a formal parameter, and trading n by $\mu = n(n+1)$, we obtain

$$\begin{aligned}L &= 4p^2 + m\left(\frac{\mu}{2} - 2p^2\right) + \frac{m^2((\mu-2)\mu - 48p^4 + 4(2\mu+3)p^2)}{32(4p^2-1)} \\ &+ \frac{m^3((\mu-2)\mu - 48p^4 + 4(2\mu+3)p^2)}{64(4p^2-1)} + \frac{m^4}{32768(p^2-1)(4p^2-1)^3} \\ &\times \left[\mu(7\mu^3 - 12\mu^2 - 332\mu + 656) - 248832p^{10} + 256(164\mu + 1701)p^8 \right. \\ &+ 64(86\mu^2 - 1148\mu - 3645)p^6 - 48(4\mu^3 + 168\mu^2 - 820\mu - 1053)p^4 \\ &\left. + 4(5\mu^4 + 24\mu^3 + 750\mu^2 - 2132\mu - 972)p^2 \right] + O(m^5).\end{aligned}\tag{4.3}$$

Of course it is necessary to keep only a finite number of terms in (4.2) for any desired order of expansion. If we now replace $p \rightarrow \nu/2$, we can see that (3.12) is reproduced by setting

$$\lambda = \left(\frac{2\mathbb{K}}{\pi}\right)^2 \left(-L + \mu \frac{m+1}{3}\right),\tag{4.4}$$

where the prefactor is the usual one converting the period from $4\mathbb{K}$ to 2π , and the shift in the round bracket is that appearing in (3.5), for generic μ . We now express m in terms of q and separate out the q -independent part according to the generalization of (3.13)

$$\lambda = \frac{\mu}{3} - \nu^2 + \Lambda,\tag{4.5}$$

we can check that (3.14) reads in general form

$$\sum_{k=1}^{\infty} \tilde{F}_k(\nu) q^{2k} = -\frac{1}{2} \int \frac{dq}{q} \Lambda(\nu, q) + 4\mu \sum_{k=1}^{\infty} \log(1 - q^{2k}).\tag{4.6}$$

For instance, the terms written in (4.3) are enough to reproduce perfectly the results (2.7). We conclude by remarking that the expansion of $\lambda(\nu)$ may also be derived by elementary means, although with some effort as briefly discussed in appendix A.

4.2 Matching poles against quasi-modular expansion

Another method to compute $\tilde{F}_k(\nu)$ is based on the general form (2.8) taking into account the structure (2.10). This is a bit involved at generic μ and we illustrate here for the simpler case of $\mu = 2$. Nevertheless, the main core of the computation will be valid for all μ . In the 1-gap case, at each instanton level, we need the $k+1$ numbers $\mathbf{d}^{(k)} = \{d_1^{(k)}, \dots, d_{k+1}^{(k)}\}$, see (2.11). These may be fixed by exploiting the quasi-modular expansion of the prepotential at large a .¹⁰ Let us briefly review the construction, see for instance [63]. We begin by looking for a solution of the Miura equation (3.3) at large λ . This takes the form

$$v = \sqrt{\lambda} + \sum_{n=1}^{\infty} \frac{v_n}{(\sqrt{\lambda})^n}. \quad (4.7)$$

Replacing (4.7) into (3.3), we immediately obtain $v_n = v_n(u, u', u'', \dots)$. The terms with even n are total derivatives. Terms with odd n involve up to the $2(n-1)$ -th derivative of u . The first cases are¹¹

$$\begin{aligned} v_1 &= \frac{1}{2} u, & v_2 &= -\frac{1}{4} u', & v_3 &= \frac{1}{8} u'' - \frac{1}{8} u^2, & v_4 &= \left(-\frac{1}{16} u'' + \frac{1}{8} u^2\right)', \\ v_5 &= \frac{1}{32} u^{(4)} - \frac{3}{16} u u'' - \frac{5}{32} u'^2 + \frac{1}{16} u^3, \dots \end{aligned} \quad (4.8)$$

The Floquet exponent $i\nu$ is obtained by taking the average of v over a period (we denote it by $\langle \dots \rangle$) giving

$$i\nu = \langle v \rangle = \sqrt{\lambda} + \sum_{n=1}^{\infty} \frac{\pi^{2n} \varepsilon_n}{(\sqrt{\lambda})^{2n-1}}. \quad (4.9)$$

Now, we use some specific properties of the the potential (3.2) suitable for the $\mathcal{N} = 2^*$ theory with gauge group $SU(2)$. We rescale the real semi-period of the Weierstrass function $\omega \rightarrow \omega = 1/2$. The Weierstrass invariants can be expressed in terms of the Eisenstein series E_4 and E_6 , see appendix C

$$g_2 = \frac{4\pi^4}{3} E_4(q), \quad g_3 = \frac{8\pi^6}{27} E_6(q). \quad (4.10)$$

The differential equation for \wp can be used to reduce v_n to combinations of powers of \wp with coefficients involving g_2, g_3 . Defining $p_n = \langle \wp^n \rangle$, we have

$$p_0 = 1, \quad p_1 = -\frac{\pi^2}{3} E_2(q), \quad (4.11)$$

¹⁰This may appear to be a backstep from our strategy of avoiding expansions in a , but we shall see later why we can use this expansion to reconstruct the full rational functions \tilde{F}_k .

¹¹The combinations v_n are basically KdV charge densities. From the general properties of the KdV equation [77], one has the following recursion $v_n = -\frac{1}{2} v'_{n-1} - \frac{1}{2} \sum_{m=1}^{n-2} v_m v_{n-m-1}$, with $v_1 = \frac{u}{2}$. This is quite direct, but has the disadvantage that one needs the trivial even charges.

and the known recursion (see appendix A.2 of [56] and also [78])¹²

$$p_n = \frac{2n-3}{4(2n-1)} g_2 p_{n-2} + \frac{n-2}{2(2n-1)} g_3 p_{n-3}. \quad (4.12)$$

In this way, we obtain the quantities ε_n in (4.9). The first cases are¹³

$$\begin{aligned} \varepsilon_1 &= -\frac{\mu}{6} E_2, & \varepsilon_2 &= -\frac{\mu^2}{72} E_4, \\ \varepsilon_3 &= -\frac{\mu^3}{2160} (9 E_2 E_4 - 4 E_6) + \frac{\mu^2}{180} (E_2 E_4 - E_6), \\ \varepsilon_4 &= -\frac{5\mu^4}{72576} (15 E_4^2 - 8 E_2 E_6) + \frac{5\mu^3}{1512} (E_4^2 - E_2 E_6) - \frac{\mu^2}{252} (E_4^2 - E_2 E_6). \end{aligned} \quad (4.13)$$

The expansion (4.9) can be inverted in the form

$$\lambda = -\nu^2 + \sum_{n=1}^{\infty} \frac{\pi^{2n} \lambda_n}{\nu^{2(n-1)}}, \quad (4.14)$$

with the explicit coefficients

$$\lambda_1 = \frac{\mu}{3} E_2, \quad \lambda_2 = \frac{\mu^2}{36} (E_2^2 - E_4), \quad \lambda_3 = \frac{\mu^3}{540} (5E_2^3 - 3E_2 E_4 - 2E_6) - \frac{\mu^2}{90} (E_2 E_4 - E_6), \quad (4.15)$$

and so on. We have computed the coefficients λ_n up to high $n \sim 50$ for later application. In the NS limit with the identification (2.5), the resummed instanton contributions to the prepotential can be expanded in inverse powers of a ¹⁴ and it takes the form¹⁵

$$F_{\text{inst}}(\epsilon_1, a) = \sum_{k=1}^{\infty} F_k(\epsilon_1, a) q^{2k} = \epsilon_1^2 f_0(q) + \sum_{n=1}^{\infty} f_n(q) \frac{\epsilon_1^{2(n+1)}}{a^{2n}}. \quad (4.16)$$

The relation between $\{f_n\}$ and $\{\lambda_n\}$ is nothing but Matone relation [79] in the present Bethe/gauge context. For $n \geq 1$, it simply reads

$$q \frac{d}{dq} f_n(q) = -\frac{1}{2^{n+1}} \lambda_{n+1}(q). \quad (4.17)$$

This equation must be interpreted in an expansion in powers of q^2 . In particular, the operation $(q \frac{d}{dq})^{-1}$ acts on the right side by replacing q^n by $\frac{1}{n} q^n$ and this fixes implicitly the integration constant in (4.17) when deriving $\{f_n\}$ from $\{\lambda_n\}$. The term f_0 can also be worked out and it is

$$f_0 = 2\mu \log \prod_{n=1}^{\infty} (1 - q^{2n}). \quad (4.18)$$

¹²The proof of (4.12) is elementary. From the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, we obtain $\wp'' = 6\wp^2 - \frac{1}{2}g_2$, and $(\wp^{n-2}\wp')' = (n-2)\wp^{n-3}(\wp')^2 + \wp^{n-2}\wp'' = 2(2n-1)\wp^n - \frac{1}{2}g_2(2n-3)\wp^{n-2} - g_3(n-2)\wp^{n-3}$. Integrating over a period, we obtain (4.12).

¹³As remarked in [66], the combinations ε_n are essentially Faulhaber polynomials [78].

¹⁴Of course, the full prepotential has also a perturbative part starting with a RG term $\sim \log(a/\Lambda)$ plus inverse powers of a . Here, we shall focus on the instantons only, *i.e.* the terms in (2.2).

¹⁵Many coefficients f_n can be found in [49] for generic m, ϵ_1 .

As is well known, it is possible to integrate (4.17) by using the differential equations obeyed by the modular functions E_2, E_4 , and E_6

$$q \frac{d}{dq} E_2(q) = \frac{1}{6} (E_2^2 - E_4), \quad q \frac{d}{dq} E_4(q) = \frac{2}{3} (E_2 E_4 - E_6), \quad q \frac{d}{dq} E_6(q) = E_2 E_6 - E_4^2. \quad (4.19)$$

One finds that f_n are quasi-modular polynomials in E_2, E_4 and E_6 of total weight $2n$. The quasi-modular part is the one involving E_2 and is governed by the modular anomaly equation expressing S-duality [49–54].

We now specialize to $\mu = 2$ and come back to the problem of determining at each instanton level the $k + 1$ numbers $\mathbf{d}^{(k)} = \{d_1^{(k)}, \dots, d_{k+1}^{(k)}\}$, see (2.11). These can be fixed by the modular expansion (4.14), using (4.16) and (4.17). In other words, we expand

$$\tilde{F}_k(\nu) = f_{0,k} + \sum_{n=1}^{\infty} 2^{2n} f_{n,k} \nu^{-2n} = f_{0,k} - \frac{1}{4k} \sum_{n=1}^{\infty} \lambda_{n+1,k} \nu^{-2n}, \quad (4.20)$$

where $f_0(q) = \sum_{k=1}^{\infty} f_{0,k} q^{2k}$ and $\lambda_n(q) = \sum_{k=1}^{\infty} \lambda_{n,k} q^{2k}$. Comparing (4.20) with the Ansatz (2.11) we determine $\mathbf{d}^{(k)}$. Just to give an example and illustrate this simple procedure, let us consider the 3-instanton case. We have

$$\tilde{F}_3(\nu) = d_1^{(3)} \left[\frac{1}{(\nu-1)^5} - \frac{1}{(\nu+1)^5} \right] + d_2^{(3)} \left[\frac{1}{(\nu-1)^3} - \frac{1}{(\nu+1)^3} \right] + d_3^{(3)} \left[\frac{1}{\nu-1} - \frac{1}{\nu+1} \right] + d_4^{(3)}. \quad (4.21)$$

Its large ν expansion is

$$\tilde{F}_3(\nu) = d_4^{(3)} + \frac{2d_3^{(3)}}{\nu^2} + \frac{6d_2^{(3)} + 2d_3^{(3)}}{\nu^4} + \frac{10d_1^{(3)} + 20d_2^{(3)} + 2d_3^{(3)}}{\nu^6} + \dots \quad (4.22)$$

From the quasi-modular expansion, we have

$$\begin{aligned} f_0(q) &= 4 \log \prod_{n=1}^{\infty} (1 - q^{2n}) = -4q^2 - 6q^4 - \frac{16q^6}{3} + \dots, \\ \lambda_2 &= \frac{1}{9} (E_2^2 - E_4) = -32q^2 - 192q^4 - 384q^6 + \dots, \\ \lambda_3 &= \frac{2}{135} (5E_2^3 - 6E_4E_2 + E_6) = -32q^2 + 192q^4 + 3456q^6 + \dots, \\ \lambda_4 &= \frac{1}{567} (35E_2^4 - 49E_4E_2^2 + 12E_6E_2 + 2E_4^2) = -32q^2 + 1088q^4 + 7296q^6 + \dots \end{aligned} \quad (4.23)$$

Hence, we must have

$$\tilde{F}_3(\nu) = -\frac{16}{3} + \frac{32}{\nu^2} - \frac{288}{\nu^4} - \frac{608}{\nu^6} + \dots \quad (4.24)$$

Comparing (4.24) and (4.22), we obtain

$$d_1^{(3)} = -\frac{128}{3}, \quad d_2^{(3)} = -\frac{160}{3}, \quad d_3^{(3)} = 16, \quad d_4^{(3)} = -\frac{16}{3}, \quad (4.25)$$

in agreement with Q_3 in (2.9).

5 The fate of Ω -deformation singularities at all instantons: finite-gap

In this section, we shall discuss the Ω -deformation singularity at $\nu = 1$ at all instantons in the 1- and 2-gap cases, i.e. at $\mu = 2$ and 6 respectively.

5.1 Full analysis of the 1-gap problem

By applying the methods discussed in the previous section, we have computed $\tilde{F}_k(\nu)$ for instanton number k up to 24. We remark that this is a non-trivial computational problem with other approaches or for general mass and/or beyond the NS limit. This effort allows to analyze possible regularities in the Nekrasov functions, or equivalently in the sets $\mathbf{d}^{(k)} = \{d_1^{(k)}, \dots, d_{k+1}^{(k)}\}$. From our explicit data we could check the following remarkable relations for the pole coefficients in (2.11). After introducing

$$\mathcal{D}_k = \frac{(-1)^{k+1} 2^{2k} (2k-2)!}{(k!)^2}, \quad (5.1)$$

we obtain

$$\begin{aligned} d_1^{(k)} &= \mathcal{D}_k, & d_2^{(k)} &= -\frac{k(2k-1)}{4(2k-3)} \mathcal{D}_k \\ d_3^{(k)} &= \frac{k(4k^3 - 18k^2 + 23k - 3)}{32(2k-3)(2k-5)} \mathcal{D}_k, \\ d_4^{(k)} &= -\frac{k(8k^5 - 96k^4 + 440k^3 - 891k^2 + 659k - 30)}{384(2k-3)(2k-5)(2k-7)} \mathcal{D}_k, \\ d_5^{(k)} &= \frac{k(16k^7 - 368k^6 + 3556k^5 - 18239k^4 + 51445k^3 - 73787k^2 + 40527k - 630)}{6144(2k-3)(2k-5)(2k-7)(2k-9)} \mathcal{D}_k, \\ d_6^{(k)} &= -\frac{k}{122800(2k-3)(2k-5)(2k-7)(2k-9)(2k-11)} \left(32k^9 - 1200k^8 + 19800k^7 \right. \\ &\quad \left. - 186330k^6 + 1082901k^5 - 3927855k^4 + 8529985k^3 - 9884655k^2 \right. \\ &\quad \left. + 4503402k - 22680 \right) \mathcal{D}_k, \end{aligned} \quad (5.2)$$

The general pattern for $p > 1$ is

$$d_p^{(k)} = \frac{(-1)^{p+1} k G_p(k)}{4^{p-1} \Gamma(p) \prod_{\ell=1}^{p-1} (2k - 2\ell - 1)} \mathcal{D}_k, \quad (5.3)$$

where $G_p(k)$ is a polynomial with degree $2p-3$ and integer coefficients. Also, it is not hard to compute additional terms beyond those in (5.2). We emphasize that these expressions are rather peculiar. They provide the exact dependence of certain features of the exact Nekrasov prepotential \tilde{F}_k in terms of formulas that are parametric in k , i.e. that hold for all k . This is quite an important fact because it opens the way to possible resummations over all instantons. This is precisely what we are going to do.

5.1.1 Summing the singular poles over all instantons

The total instanton partition function is

$$\tilde{F}(\nu, q) = \sum_{k=1}^{\infty} \tilde{F}_k(\nu) q^{2k}. \quad (5.4)$$

As a first step, we analyze its singular part in the $\nu \rightarrow 1$ limit. From (2.11), this is

$$\begin{aligned} \tilde{F}^{\text{sing}}(\nu, q) &= \sum_{k=1}^{\infty} \sum_{p=1}^k \frac{d_p^{(k)} q^{2k}}{(\nu-1)^{2k-2p+1}} = \sum_{p=1}^{\infty} \sum_{k=p}^{\infty} \frac{d_p^{(k)} q^{2k}}{(\nu-1)^{2k-2p+1}} \\ &= \sum_{p=1}^{\infty} q^{2p-1} \sum_{k=p}^{\infty} \frac{d_p^{(k)} q^{2k-2p+1}}{(\nu-1)^{2k-2p+1}} = \sum_{k=1}^{\infty} q^{2k-1} \sum_{p=0}^{\infty} d_k^{(k+p)} \left(\frac{q}{\nu-1} \right)^{2p+1} \\ &= \sum_{k=1}^{\infty} g_k \left(\frac{q}{\nu-1} \right) q^{2k-1}, \end{aligned} \quad (5.5)$$

where we have introduced the functions

$$g_k(z) = \sum_{p=0}^{\infty} d_k^{(k+p)} z^{2p+1}. \quad (5.6)$$

Let us look at the term with $k = 1$. This is not the pure 1-instanton term because the argument of the functions g_k have an argument that also depends on q . An explicit summation gives g_1 in terms of an hypergeometric function that can be expressed by elementary functions

$$g_1(z) = 4z {}_3F_2 \left(\frac{1}{2}, 1, 1; 2, 2; -16z^2 \right) = \frac{\sqrt{16z^2+1} - \log \left[\frac{e}{2} (\sqrt{16z^2+1} + 1) \right]}{z} \quad (5.7)$$

We can now take the $z \rightarrow +\infty$ limit and we find

$$g_1(z) = 4 + \mathcal{O}(1/z), \quad (5.8)$$

where the omitted terms have also logarithmically enhanced term $\sim (\log z)/z$. A similar result is obtained for the other functions $g_k(z)$. In particular, we find

$$\begin{aligned} g_2(z) &= \frac{32\sqrt{16z^2+1}z^2 - 24z^2 - \sqrt{16z^2+1} + 1}{24z^3} = \frac{16}{3} + \mathcal{O}(1/z), \\ g_3(z) &= \frac{-480z^4 - 80z^2 + 1}{320z^5} + \frac{6144z^6 + 1152z^4 + 72z^2 - 1}{320z^5\sqrt{16z^2+1}} = \frac{24}{5} + \mathcal{O}(1/z), \\ g_4(z) &= \frac{7168z^6 + 1344z^4 - 168z^2 + 1}{2688z^7} \\ &\quad + \frac{786432z^{10} - 270336z^8 - 23040z^6 + 2592z^4 + 144z^2 - 1}{2688z^7(16z^2+1)^{3/2}} = \frac{32}{7} + \mathcal{O}(1/z), \\ g_5(z) &= \frac{-87552z^8 + 27648z^6 + 8640z^4 - 288z^2 + 1}{18432z^9} \\ &\quad + \frac{1}{18432z^9(16z^2+1)^{5/2}} \left(109051904z^{14} + 38273024z^{12} - 21135360z^{10} \right. \\ &\quad \left. - 4794368z^8 - 236288z^6 + 2400z^4 + 248z^2 - 1 \right) = \frac{52}{9} + \mathcal{O}(1/z). \end{aligned} \quad (5.9)$$

Collecting these results and adding similar computations for $g_6(z)$ and $g_7(z)$ we get

$$\tilde{F}^{\text{sing}}(1, q) = 4q + \frac{16}{3}q^3 + \frac{24}{5}q^5 + \frac{32}{7}q^7 + \frac{52}{9}q^9 + \frac{48}{11}q^{11} + \frac{56}{13}q^{13} + \dots \quad (5.10)$$

Remarkably, this expression takes the form of a *finite* series in q .¹⁶ Besides, one can check that the expansion (5.10) agrees with

$$\tilde{F}^{\text{sing}}(1, q) = 2 \log \frac{\prod_{n=1}^{\infty} (1 - (-q)^n)}{\prod_{n=1}^{\infty} (1 - q^n)} = -\frac{1}{4} \log(1 - m), \quad m = m(q). \quad (5.11)$$

Actually, (5.11) is not a guess, but has a natural origin from the Lamé equation as we shall discuss in a moment. Here, we just add that using the known relation between m and q ,

$$\frac{dq}{dm} = \frac{\pi^2 q}{4m(1-m)\mathbb{K}^2}, \quad (5.12)$$

it is possible to rewrite the $q \partial_q$ derivative of the singular part (5.11) in the more suggestive form

$$q \frac{d}{dq} \tilde{F}^{\text{sing}}(1, q) = \frac{m}{4} \left(\frac{2\mathbb{K}}{\pi} \right)^2. \quad (5.13)$$

5.1.2 Summing the regular part over all instantons

A completely similar computation can be repeated for the part of (5.4) that is regular when $\nu \rightarrow 1$. This is closely related to the singular part due to the exact partial fraction decomposition (2.11). Apart from the $d_{k+1}^{(k)}$ constant term, the contributions proportional to powers of $1/(\nu + 1)$ give, see (5.5) and (4.18)

$$\begin{aligned} \tilde{F}^{\text{reg}}(1, q) &= \tilde{F}^{\text{reg(I)}}(q) + \tilde{F}^{\text{reg(II)}}(q), \\ \tilde{F}^{\text{reg(I)}}(q) &= - \sum_{k=1}^{\infty} g_k \left(\frac{q}{2} \right) q^{2k-1}, \quad \tilde{F}^{\text{reg(II)}}(q) = 4 \log \prod_{n=1}^{\infty} (1 - q^{2n}). \end{aligned} \quad (5.14)$$

The explicit computation of the functions $g_k(z)$ that we did in (5.7) and (5.9) allows to evaluate the non-trivial part of (5.14) as follows

$$\begin{aligned} \tilde{F}^{\text{reg(I)}}(q) &= -q \left(2q - q^3 + \frac{4q^5}{3} - \frac{5q^7}{2} + \frac{28q^9}{5} + \dots \right) \\ &\quad - q^3 \left(6q - \frac{20q^3}{3} + 14q^5 - 36q^7 + \dots \right) - q^5 \left(8q - 19q^3 + \frac{324q^5}{5} + \dots \right) \\ &\quad - q^7 \left(14q - 44q^3 + \dots \right) - q^9 \left(12q + \dots \right) + \mathcal{O}(q^{12}) \\ &= -2q^2 - 5q^4 - \frac{8q^6}{3} - \frac{13q^8}{2} - \frac{12q^{10}}{5} + \mathcal{O}(q^{12}). \end{aligned} \quad (5.15)$$

¹⁶The fact that (5.10) is organized in *odd* powers of q is not in contradiction with (5.4). The Taylor expansion of the functions $g_k(z)$ around $z = 0$ involves odd powers of z , see (5.6), but here we are taking the large $z \rightarrow +\infty$ expansion.

Adding the second term in (5.14), the full regular part of $\tilde{F}(\nu)$ at $\nu = 1$ is thus

$$\tilde{F}^{\text{reg}}(1, q) = -6q^2 - 11q^4 - 8q^6 - \frac{27q^8}{2} - \frac{36q^{10}}{5} - \frac{44q^{12}}{3} - \frac{48q^{14}}{7} - \frac{59q^{16}}{4} - \frac{26q^{18}}{3} + \dots, \quad (5.16)$$

where we have included a few additional terms beyond those written in (5.15).

5.1.3 All instanton partition function at $\nu = 1$ from the Lamé equation

At this point we can understand what is happening. Summing the singular poles and the regular part of $\tilde{F}(1)$ we obtain from (5.10) and (5.16) the total result

$$\begin{aligned} \tilde{F}(1, q) = & 4q - 6q^2 + \frac{16q^3}{3} - 11q^4 + \frac{24q^5}{5} - 8q^6 + \frac{32q^7}{7} - \frac{27q^8}{2} + \frac{52q^9}{9} - \frac{36q^{10}}{5} \\ & + \frac{48q^{11}}{11} - \frac{44q^{12}}{3} + \frac{56q^{13}}{13} - \frac{48q^{14}}{7} + \frac{32q^{15}}{5} - \frac{59q^{16}}{4} + \frac{72q^{17}}{17} - \frac{26q^{18}}{3} + \dots \end{aligned} \quad (5.17)$$

Comparing with (3.14), this means

$$\begin{aligned} \Lambda(\nu = 1, q) = & -2q \frac{d}{dq} \left[\tilde{F}(1, q) - 8 \sum_{n=1}^{\infty} \log(1 - q^{2n}) \right] = -8q - 8q^2 - 32q^3 - 8q^4 - 48q^5 \\ & - 32q^6 - 64q^7 - 8q^8 - 104q^9 - 48q^{10} - 96q^{11} - 32q^{12} - 112q^{13} \\ & - 64q^{14} - 192q^{15} - 8q^{16} - 144q^{17} - 104q^{18} + \mathcal{O}(q^{19}). \end{aligned} \quad (5.18)$$

However, the exact spectrum of the Lamé equation predicts that $\Lambda(\nu = 1, q)$ is nothing but $\lambda + \frac{1}{3}$ where λ is the eigenvalue of the scattering band edge associated with the potential (3.5). For the potential $V(x) = 2m \operatorname{sn}^2(x, m)$ this is simply $m + 1$. Taking into account the sign of the eigenvalue as defined in (4.7), as well as the scaling factor to map the equation on $x \in [0, 2\pi]$, we have the exact result

$$\Lambda(\nu = 1, q) = \frac{1}{3} - \left[2 \left(-\frac{m+1}{3} \right) + m + 1 \right] \left(\frac{2\mathbb{K}}{\pi} \right)^2 = \frac{1}{3} - \frac{m+1}{3} \left(\frac{2\mathbb{K}}{\pi} \right)^2 \quad (5.19)$$

The expansion of this quantity gives indeed

$$\frac{1}{3} - \frac{m+1}{3} \left(\frac{2\mathbb{K}}{\pi} \right)^2 = -\frac{m}{2} - \frac{9m^2}{32} - \frac{13m^3}{64} - \frac{1321m^4}{8192} - \frac{2207m^5}{16384} - \frac{30461m^6}{262144} + \dots, \quad (5.20)$$

and replacing m by q using (3.4) we recover precisely (5.18). This agreement is a check of our calculations and, in particular, of the closed expressions in (5.2).

5.2 Singular poles for the 2-gap problem

A similar analysis can be worked out at the 2-gap point $\mu = 2 \times 3 = 6$. In this case, the partial fraction expansion (2.11) turns out to be just slightly more complicated and reads

$$\begin{aligned} \tilde{F}_k(\nu) = & d_{k+1}^{(k)} + \sum_{p=1}^k d_p^{(k)} \left(\frac{1}{(\nu-1)^{2k-2p+1}} - \frac{1}{(\nu+1)^{2k-2p+1}} \right) \\ & + \sum_{p=1}^{\lfloor \frac{k}{2} \rfloor} c_p^{(k)} \left(\frac{1}{(\nu-2)^{2\lfloor \frac{k}{2} \rfloor - 2p+1}} - \frac{1}{(\nu+2)^{2\lfloor \frac{k}{2} \rfloor - 2p+1}} \right). \end{aligned} \quad (5.21)$$

This reflects the fact that there is a new set of Nekrasov poles located at $\nu = \pm 2$. Other poles are always absent reflecting the special finite-gap choice. We can repeat the analysis we did for the 1-gap case. In particular, it is interesting to study the behaviour of the singularity at the previous point $\nu = 1$ that is still singular. Again, it is possible to derive closed expressions for the coefficients $d_p^{(k)}$. The explicit results for $\mu = 6$ (replacing the $\mu = 2$ results in (5.2)) are now similar, but slightly different

$$\begin{aligned} d_1^{(k)} &= \mathcal{D}_k, & d_2^{(k)} &= \frac{k(14k-23)}{36(2k-3)} \mathcal{D}_k \\ d_3^{(k)} &= \frac{k(196k^3 + 430k^2 - 3737k + 3597)}{2592(2k-3)(2k-5)} \mathcal{D}_k, \\ d_4^{(k)} &= \frac{k(2744k^5 + 31584k^4 - 171640k^3 - 287133k^2 + 1734725k - 1375890)}{279936(2k-3)(2k-5)(2k-7)} \mathcal{D}_k, \end{aligned} \quad (5.22)$$

and so on, where now

$$\mu = 6 : \quad \mathcal{D}_k = -\frac{(-9)^k 4^{2k-1} \Gamma(k - \frac{1}{2})}{\sqrt{\pi} k^2 \Gamma(k)}. \quad (5.23)$$

The singular part of the full instanton partition function turns out to be compatible with the Ansatz

$$\mu = 6 : \quad \tilde{F}^{\text{sing}}(1) = -\frac{3}{4} \log(1-m). \quad (5.24)$$

Thus, apart from a μ -dependent normalization, this singular part has the same dependence on m as in the 1-gap case. In the next section, we shall explain that this is an accidental fact and provide the general case. We remark that the finiteness of $\tilde{F}^{\text{sing}}(1)$ is ultimately due to the finiteness of the Lamé eigenvalues in terms of the quasi-momentum, on the edges of the spectral gaps. This is essentially the function $\Lambda(\nu, q)$. However, $\tilde{F}^{\text{sing}}(1)$ is only a part of it, because Λ includes the contributions from the other poles as well as the regular parts.

6 Beyond finite-gap, generic μ

The pattern in (5.1), (5.2) and (5.22), (5.23) suggests a universal structure in k with the fine details being dependent on the parameter μ . Guided by this results, we analyzed the $\nu = 1$ pole at generic μ . Writing again

$$\tilde{F}_k(\nu) = \sum_{p=1}^k \frac{d_p^k}{(\nu-1)^{2k-2p+1}} + \dots, \quad (6.1)$$

After some work, we confirm that the pattern is indeed rather simple. Indeed, for generic μ we obtain

$$\begin{aligned} \tilde{F}_k^{\text{sing}}(\nu) &= \frac{(-4)^{k-1} \mu^{2k} \Gamma(k - \frac{1}{2})}{\sqrt{\pi} k^2 \Gamma(k)} \left\{ \frac{1}{(\nu-1)^{2k-1}} \right. \\ &+ \frac{k[k(3\mu^2 - 8\mu - 4) - 4(\mu^2 - 2\mu - 1)]}{4\mu^2(2k-3)} \frac{1}{(\nu-1)^{2k-3}} \end{aligned} \quad (6.2)$$

$$\begin{aligned}
 & + \frac{k}{576 \mu^4 (2k-3)(2k-5)} [18k^3 (3\mu^2 - 8\mu - 4)^2 + k^2 (-289\mu^4 + 2992\mu^3 - 3208\mu^2 \\
 & - 5184\mu - 1296) + k (-629\mu^4 - 2128\mu^3 + 4216\mu^2 + 7488\mu + 1872) \\
 & + 864(\mu^4 - 2\mu^2 - 4\mu - 1)] \frac{1}{(\nu-1)^{2k-5}} + \dots \Big\},
 \end{aligned}$$

and so on. Of course, this expression reproduces the special 1- and 2-gap cases. However, (6.2) shows that the all-instanton structure at generic μ is essentially the same. Evaluating $\tilde{F}^{\text{sing}}(1, q)$, we get the remarkably simple expansion¹⁷

$$\begin{aligned}
 \tilde{F}^{\text{sing}}(1, q) &= 2\mu q - \frac{\mu}{6} (\mu^2 - 8\mu - 4) q^3 \\
 &+ \frac{\mu}{360} (11\mu^4 - 80\mu^3 + 8\mu^2 + 576\mu + 144) q^5 \\
 &+ \frac{\mu}{8064} (55\mu^6 - 1232\mu^5 + 8568\mu^4 - 21824\mu^3 + 15472\mu^2 + 13824\mu + 2304) q^7 + \dots
 \end{aligned} \tag{6.3}$$

In particular, at the n -gap points, we find¹⁸

$$\begin{aligned}
 \text{1-gap} \quad \tilde{F}^{\text{sing}}(1, q) &= 4q + \frac{16}{3} q^3 + \frac{24}{5} q^5 + \frac{32}{7} q^7 + \dots, \\
 \text{2-gap} \quad \tilde{F}^{\text{sing}}(1, q) &= 12q + 16q^3 + \frac{72}{5} q^5 + \frac{96}{7} q^7 + \dots, \\
 \text{3-gap} \quad \tilde{F}^{\text{sing}}(1, q) &= 24q - 88q^3 + \frac{16344}{5} q^5 + \frac{192}{7} q^7 + \dots, \\
 \text{4-gap} \quad \tilde{F}^{\text{sing}}(1, q) &= 40q - \frac{2360}{3} q^3 + 63048 q^5 + \frac{13547840}{7} q^7 + \dots
 \end{aligned} \tag{6.4}$$

Actually, we now show that the expansion (6.3) has a very simple interpretation from the point of view of the spectrum of the Lamé equation. This is most clearly explained resorting again to the n -gap cases. The bands for the potential $V(x) = n(n+1)m \operatorname{sn}^2(x, m)$ are for $n = 1, 2, 3$

$$\begin{aligned}
 \text{1-gap} \quad & [E_0^{(1)}, E_1^{(1)}] \cup [E_2^{(1)}, +\infty), \\
 \text{2-gap} \quad & [E_0^{(2)}, E_1^{(2)}] \cup [E_2^{(2)}, E_3^{(2)}] \cup [E_4^{(2)}, +\infty), \\
 \text{3-gap} \quad & [E_0^{(3)}, E_1^{(3)}] \cup [E_2^{(3)}, E_3^{(3)}] \cup [E_4^{(3)}, E_5^{(3)}] \cup [E_6^{(3)}, +\infty).
 \end{aligned} \tag{6.5}$$

The relevant edges for our discussion are the upper edge of the first bound band and the lower edge of the second bound (or scattering in 1-gap case) band. This is because we are looking at the singularities at $\nu = 1$. The difference $E_2^{(n)} - E_1^{(n)}$ is the *width of the first gap*. For $n = 1, 2, 3$, it is known that

$$\begin{aligned}
 E_2^{(1)} - E_1^{(1)} &= m, & E_2^{(2)} - E_1^{(2)} &= 3m, \\
 E_2^{(3)} - E_1^{(3)} &= 2\sqrt{m^2 - m + 4} - 2\sqrt{4m^2 - 7m + 4} + 3m.
 \end{aligned} \tag{6.6}$$

¹⁷We stress again that $\tilde{F}^{\text{sing}}(1, q)$ is obtained from $\tilde{F}^{\text{sing}}(\nu, q) = \sum_k \tilde{F}_k^{\text{sing}}(\nu) q^{2k}$, doing the sum over k , i.e. over all instanton, and taking the limit $\nu \rightarrow 1^+$.

¹⁸As we remarked, the 2-gap result is 3 times the 1-gap one. This accidental simplification will find its explanation in the relation (6.7).

Comparing the first line of (6.6) with (5.13) and (5.24), it is natural to conjecture that

$$q \frac{d}{dq} \tilde{F}^{\text{sing}}(1, q) = \frac{E_2^{(n)} - E_1^{(n)}}{4} \left(\frac{2\mathbb{K}}{\pi} \right)^2. \quad (6.7)$$

Indeed and rather non-trivially, this also agrees with the 3-gap case. In particular, it implies the following all-order expansion replacing the third row of (6.4)

$$\begin{aligned} \text{3-gap} \quad \tilde{F}^{\text{sing}}(1, q) = & 24q - 88q^3 + \frac{16344}{5}q^5 + \frac{192}{7}q^7 - \frac{27856816}{3}q^9 \\ & + \frac{7624002168}{11}q^{11} - \frac{122964074664}{13}q^{13} - \frac{16297316089728}{5}q^{15} \\ & + \frac{6101221040271792}{17}q^{17} - \frac{202648353741486600}{19}q^{19} + \dots \end{aligned} \quad (6.8)$$

Actually, a similar discussion can be done for generic μ . To this aim, we need the expansion of the Lamé eigenvalues E_1, E_2 for generic μ . This can be achieved efficiently by the methods presented in [76]. We did this exercise at high order in the modular parameter m . The first terms of the expansions are

$$\begin{aligned} E_1 = & 1 + \frac{1}{4}(\mu - 2)m - \frac{1}{128}((\mu - 6)(\mu - 2))m^2 + \frac{(\mu - 16)(\mu - 6)(\mu - 2)m^3}{4096} \\ & - \frac{((\mu - 6)(\mu - 2)(\mu^2 - 104\mu + 972))m^4}{393216} \\ & - \frac{((\mu - 6)(\mu - 2)(11\mu^3 + 152\mu^2 - 9420\mu + 66240))m^5}{37748736} \\ & + \frac{(\mu - 6)(\mu - 2)(49\mu^4 - 1608\mu^3 - 11668\mu^2 + 552960\mu - 3223296)m^6}{2415919104} + \dots, \\ E_2 = & 1 + \frac{1}{4}(3\mu - 2)m - \frac{1}{128}((\mu - 6)(\mu - 2))m^2 - \frac{((\mu - 6)(\mu - 2)(\mu + 16))m^3}{4096} \\ & - \frac{((\mu - 6)(\mu - 2)(\mu^2 + 88\mu + 972))m^4}{393216} \\ & + \frac{(\mu - 6)(\mu - 2)(11\mu^3 - 136\mu^2 - 7116\mu - 66240)m^5}{37748736} \\ & + \frac{(\mu - 6)(\mu - 2)(49\mu^4 + 1208\mu^3 - 9620\mu^2 - 384000\mu - 3223296)m^6}{2415919104} + \dots \end{aligned} \quad (6.9)$$

Their difference is somewhat simpler

$$\begin{aligned} E_2 - E_1 = & \frac{\mu m}{2} - \frac{(\mu - 6)(\mu - 2)\mu m^3}{2048} - \frac{(\mu - 6)(\mu - 2)\mu m^4}{2048} \\ & + \frac{(\mu - 6)(\mu - 2)\mu(11\mu^2 + 8\mu - 8268)m^5}{18874368} + \frac{(\mu - 6)(\mu - 2)\mu(11\mu^2 + 8\mu - 3660)m^6}{9437184} + \dots \end{aligned} \quad (6.10)$$

Using (6.10) and (6.3), we match perfectly (6.7) at the considered order. This provides a non-trivial check that the relation (6.7) holds for any μ .

Actually, it seems quite natural to generalize (6.7) to the following more general form

$$q \frac{d}{dq} \tilde{F}^{\text{sing}}(N, q) = \frac{W_N}{4} \left(\frac{2\mathbb{K}}{\pi} \right)^2, \quad (6.11)$$

where $N = 1, 2, \dots$, and W_N is the width of the N -th gap of the Lamé spectrum. A simple check of (6.11) in the 2-gap case is indeed presented in appendix B.

7 Conclusions

To summarize, we have shown that the k -instanton prepotential of $\mathcal{N} = 2^* \text{SU}(2)$ gauge theory has a nice structure in the n -gap Nekrasov-Shatashvili limit defined by

$$\epsilon_2 = 0, \quad m = \left(n + \frac{1}{2}\right) \epsilon_1, \quad n \in \mathbb{N}. \quad (7.1)$$

Up to trivial factors, it is a rational function of the ratio $\nu = \frac{2a}{\epsilon_1}$ where a is the scalar field expectation value. This Nekrasov function $\tilde{F}_k(\nu)$ has poles at $|\nu| = 1, 2, \dots, n$ only. We studied the structure of this poles (and of the regular part) by exploiting the Bethe/gauge map finding simple closed expressions parametric in k .

The Bethe/gauge correspondence predicts that the Nekrasov functions may be obtained from the eigenvalues λ of the associated spectral problem for the Lamé equation. What is needed is simply the expression of λ as a function of the Floquet exponent to be identified with the above ratio ν . The exact expression $\lambda(\nu)$ is not singular at the Nekrasov poles $|\nu| = 1, 2, \dots, n$. The poles are an artifact of the modular expansion in powers of the instanton counting coupling q^2 . We explicitly clarified these relations. In particular, in the 1-gap case, we summed over all instantons and showed explicitly the cancellation of the Nekrasov poles. Adding the finite part of the Nekrasov functions, we also checked agreement with the Lamé eigenvalue at the scattering band edge.

Focusing on the singular part only, we remark that the all-instanton finite resummation of the Nekrasov poles is a special contribution to the total Nekrasov function, not immediately related to quantities appearing in the Lamé problem. As we explained in the Introduction, it is interesting because it is expected to display some simple behaviour. This is due to the fact that, at any instanton number, the poles are associated with the structure of supersymmetric states in a quantum mechanical model that arises in the 5d Ω -deformation construction. Actually, we showed that the resummed poles give a finite contribution at generic m/ϵ_1 — not necessarily finite-gap — and provided its clean Lamé equation interpretation in terms of the widths of the spectral gaps.

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A Direct perturbative expansion of $\lambda(\nu)$ from the Lamé equation

The expansion (3.11) may be also obtained by a direct perturbative approach [80]. Just to show how it works, one simply writes the Lamé equation with rescaled variable $x \in [0, 2\pi]$

and writes

$$\begin{aligned} \psi''(x) - 2 \left[-\frac{m+1}{3} + m \operatorname{sn}^2 \left(\frac{2\mathbb{K}}{\pi} x, m \right) \right] \psi(x) &= \lambda \psi(x), \\ \lambda &= \frac{2}{3} - \nu^2 + \lambda_1 m + \lambda_2 m^2 + \lambda_3 m^3 + \dots \end{aligned} \quad (\text{A.1})$$

The $\mathcal{O}(m^3)$ wave function with definite quasi-momentum may be found by some calculation and it is

$$\begin{aligned} \psi(x) = e^{i\nu x} & \left[1 - \frac{m}{2} + \frac{i\nu m}{4(\nu^2 - 1)} \left(1 + \frac{21\nu^4 - 46\nu^2 + 9}{256(\nu^2 - 1)^2} m^2 + \right) \sin(2x) \right. \\ & - \frac{m}{4(\nu^2 - 1)} \left(1 + \frac{23\nu^4 - 58\nu^2 + 19}{256(\nu^2 - 1)^2} m^2 \right) \cos(2x) \\ & + \frac{i\nu m^2}{64(\nu^2 - 1)} \left(1 + \frac{m}{2} \right) \sin(4x) \\ & - \frac{m^2}{64(\nu^2 - 1)} \left(1 + \frac{m}{2} \right) \cos(4x) + \frac{i\nu m^3}{1024(\nu^2 - 1)} \sin(6x) \\ & \left. - \frac{m^3}{1024(\nu^2 - 1)} \cos(6x) + \mathcal{O}(m^4) \right]. \end{aligned} \quad (\text{A.2})$$

Replacing (A.2) in (A.1) determines $\lambda_1, \lambda_2, \lambda_3$ in agreement with (3.11). The structure is clear, but going to very high orders in m is rather cumbersome, whereas the approach described in the main text and based on the representation (3.9) is fully straightforward and bypass the construction of $\psi(x)$.

B Resummation of singular 2-gap poles at $\nu = 2$

At $\mu = 2 \times 3 = 6$, the general structure of Nekrasov functions is (5.21). An analysis of the terms proportional to inverse powers of $\nu \pm 2$ gives for $k \geq 1$

$$\begin{aligned} \text{even } k : \quad c_1^{(k)} = \mathcal{D}_k^{(+)} &\equiv \frac{(-1)^{\frac{k}{2}+1} 24^k \Gamma(\frac{k-1}{2})}{\sqrt{\pi} k^2 \Gamma(\frac{k}{2})}, \\ \text{odd } k : \quad c_1^{(k)} = \mathcal{D}_k^{(-)} &\equiv \frac{(-1)^{\frac{k-1}{2}} 3^{k-1} 2^{3k+1} \Gamma(\frac{k}{2} - 1)}{\sqrt{\pi} \Gamma(\frac{k+1}{2})}. \end{aligned} \quad (\text{B.1})$$

At next-to-leading order, we obtain the following rational corrections for $k \geq 4$

$$\begin{aligned} \text{even } k : \quad c_2^{(k)} &= -\mathcal{D}_k^{(+)} \frac{k(256k^2 + 473k + 119)}{288(k-3)}, \\ \text{odd } k : \quad c_2^{(k)} &= -\mathcal{D}_k^{(-)} \frac{256k^3 + 1163k^2 + 941k + 520}{864(k-4)}. \end{aligned} \quad (\text{B.2})$$

The pattern continues in the same way. For $k \geq 6$,

$$\text{even } k : \quad c_3^{(k)} = \mathcal{D}_k^{(+)} \frac{k(65536k^5 + 595456k^4 + 1451987k^3 + 1929413k^2 + 1572789k + 367182)}{497664(k-5)(k-3)}$$

odd k : (B.3)

$$c_3^{(k)} = \mathcal{D}_k^{(-)} \frac{65536k^6 + 948736k^5 + 4141343k^4 + 9278505k^3 + 14843229k^2 + 9404819k + 3250632}{2488320(k-6)(k-4)}$$

The resummation of the poles is then

$$\sum_{\substack{k=2 \\ \text{even } k}}^{\infty} c_1^{(k)} \frac{q^{2k}}{(\nu-2)^{k-1}} = \frac{144q^4 {}_3F_2(\frac{1}{2}, 1, 1; 2, 2; -\frac{576q^4}{(\nu-2)^2})}{\nu-2} = 24q^2 + \mathcal{O}(\nu-2),$$

$$\sum_{\substack{k=3 \\ \text{odd } k}}^{\infty} c_1^{(k)} \frac{q^{2k}}{(\nu-2)^{k-2}} = -768q^4 + \mathcal{O}(\nu-2), \quad (\text{B.4})$$

$$\sum_{\substack{k=4 \\ \text{even } k}}^{\infty} c_2^{(k)} \frac{q^{2k}}{(\nu-2)^{k-3}} = 61472q^6 + \mathcal{O}(\nu-2),$$

$$\sum_{\substack{k=5 \\ \text{odd } k}}^{\infty} c_2^{(k)} \frac{q^{2k}}{(\nu-2)^{k-4}} = -6703104q^8 + \mathcal{O}(\nu-2), \quad (\text{B.5})$$

$$\sum_{\substack{k=6 \\ \text{even } k}}^{\infty} c_3^{(k)} \frac{q^{2k}}{(\nu-2)^{k-5}} = \frac{4346265744}{5}q^{10} + \mathcal{O}(\nu-2),$$

$$\sum_{\substack{k=7 \\ \text{odd } k}}^{\infty} c_3^{(k)} \frac{q^{2k}}{(\nu-2)^{k-6}} = -125613343744q^{12} + \mathcal{O}(\nu-2), \quad (\text{B.6})$$

and so on. The total singular part at $\nu = 2$ is therefore

$$\begin{aligned} \tilde{F}^{\text{sing}}(\nu=2) &= 24q^2 - 768q^4 + 61472q^6 - 6703104q^8 \\ &\quad + \frac{4346265744}{5}q^{10} - 125613343744q^{12} + \dots \end{aligned} \quad (\text{B.7})$$

Applying $q\partial_q$ and expressing the result in terms of the modular parameter m we get

$$\begin{aligned} q\partial_q \tilde{F}^{\text{sing}}(\nu=2) &= \frac{3m^2}{16} + \frac{3m^3}{16} + \frac{63m^4}{512} + \frac{15m^5}{256} + \frac{3585m^6}{131072} + \frac{3843m^7}{131072} + \frac{183711m^8}{4194304} \\ &\quad + \frac{52527m^9}{1048576} + \frac{360024561m^{10}}{8589934592} + \frac{235184565m^{11}}{8589934592} \\ &\quad + \frac{5020080441m^{12}}{274877906944} + \frac{2640895803m^{13}}{137438953472} + \mathcal{O}(m^{14}). \end{aligned} \quad (\text{B.8})$$

The width of the second gap of the Lamé equation at $\mu = 6$ is

$$W_2 = 2\sqrt{m^2 - m + 1} + m - 2. \quad (\text{B.9})$$

One checks that (6.11) holds, i.e.

$$q\partial_q \tilde{F}^{\text{sing}}(\nu=2) = \frac{1}{4} W_2 \left(\frac{2\mathbb{K}}{\pi} \right)^2. \quad (\text{B.10})$$

C Modular functions

The Eisenstein series appearing in the main text may be defined by the expansions

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \quad E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \quad (\text{C.1})$$

The series E_4 and E_6 are true modular forms of weight 4 and 6. Under $\text{SL}(2, \mathbb{Z})$ modular transformation

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad \text{with } ad - bc = 1, \quad (\text{C.2})$$

they transform as

$$E_4(\tau') = (c\tau + d)^4 E_4(\tau), \quad E_6(\tau') = (c\tau + d)^6 E_6(\tau). \quad (\text{C.3})$$

The series $E_2(q)$ is a quasi-modular form of weight 2, with the transformation property

$$E_2(\tau') = (c\tau + d)^2 E_2(\tau) + \frac{6}{i\pi} c(c\tau + d). \quad (\text{C.4})$$

We also remind the definition of the Dedekind η -function

$$\eta(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}). \quad (\text{C.5})$$

Its modular properties may be found, for instance, in [81].

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